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An alternative proof of the formula for the average workload cost for the D-policy in the M/G/1 queue

H.C. Tijms

ABSTRACT. This paper presents an alternative derivation of the formula for the average workload cost for the D-policy in the M/G/1 queue with a linear cost structure. The D-policy turns the server off only when the system is empty and turns the server on when the workload exceeds the value D. The approach used in this paper may be itself of interest.

Consider a single-server queueing system in which jobs arrive in accordance with a Poisson process with rate λ. The service times of the jobs are known upon arrival and are independently sampled from a distribution having probability distribution function $F(\cdot)$ with finite first moment μ and finite second moment $\mu^{(2)}$. Assume that $F(0)=0$ and $\rho<1$, where $\rho=\lambda\mu$. The workload at epoch t, $V(t)$, is defined as the sum of the service times of all jobs queueing at epoch t plus the remaining service time of the job being served at epoch t, $t\geq 0$. The system is controlled by the D-policy which turns the server off only when the system is empty and turns the server on when the workload exceeds D. When turned on the server provides service where the order of service is unimportant assuming that $V(t)$ is independent of this order. The following costs are considered. There is a cost of $K\geq 0$ for turning the server off (any cost for turning the server on may be included in K) and there is a holding cost of $h>0$ per unit workload per unit time.

A formula for the average cost was found by BALACHANDRAN[1]; see also TIJMS[8] where this formula was corrected and simplified. The derivation of this formula was based on a standard result from the theory of renewal reward processes. In this paper we give an alternative derivation. The more general approach used in this paper was suggested by DE LEVE's[3] treatment of general Markovian decision processes. First we study a Markov chain imbedded at points in time where either the server is turned off or is left off. Next we show that the average cost can be found from the stationary distribution of this Markov chain and simple cost and time functions having intuitive explanations. This approach may be itself of interest and seems rather widely applicable to controlled inventory and queueing systems (cf. chapter 4 in Tijms[7]).

PRELIMINARIES

We define first the following random variables for the 0-policy. Given that $V(0)=x$, let $T(x)=\inf\{t\geq 0 \mid V(t)=0\}$, and let $W(x)=\int_0^{T(x)} V(s)ds$. That is, $T(x)$ is the first epoch at which the system is empty and $W(x)$ is the cumulative workload up to that epoch when the server is always on.

LEMMA 1. *$ET(x) = x/(1-\rho)$ and $EW(x) = x^2/2(1-\rho) + \lambda x\mu^{(2)}/2(1-\rho)^2$ for $x\geq 0$.*

PROOF. For completeness we give a simple proof of the second relation; the proof of the first one is identical. Let $A(x)$ be the number of jobs arriving in $(0,x)$, $x>0$. Given that $A(x)=n$ each of the n arrival epochs has expectation $x/2$ (e.g. p. 17 in ROSS[6]). Let $t_b = \int_0^\infty ET(y)dF(y)$, and let $w_b = \int_0^\infty EW(y)dF(y)$. Then, for any $x>0$,

$$E(W(x) \mid A(x)=n) = x^2/2 + n(x/2)\mu + nw_b + \textstyle\sum_{k=1}^{n}(n-k)t_b\mu.$$

By unconditioning on $A(x)$ and next integrating with respect to $F(x)$, we get the formulae for w_b and $EW(x)$.

The following functions will appear to be important. For any $x\geq 0$, let

$$k_0(x) = hEW(x), \quad k_1(x) = K(1-\delta(x)) + hx/\lambda + h\int_0^\infty EW(x+u)dF(u),$$
$$t_0(x) = ET(x), \quad t_1(x) = 1/\lambda + \int_0^\infty ET(x+u)dF(u),$$

where $\delta(0) = 0$ and $\delta(x) = 1$ for $x>0$. By Lemma 1, for any $x\geq 0$,

$$k_1(x)-k_0(x)=K(1-\delta(x))+hx/\lambda(1-\rho)+h\mu^{(2)}/2(1-\rho)^2, \quad t_1(x)-t_0(x)=1/\lambda(1-\rho). \qquad (1)$$

For the situation where the workload equals $x>0$ and the server is off $k_1(x)-k_0(x)$ gives the difference between the expected holding cost incurred until the system is empty for turning the server on immediately and for turning the server on at the next arrival. A similar interpretation holds for $t_1(x)-t_0(x)$.

For any $x\geq 0$, let $M(x) = \sum_{n=1}^\infty F^n(x)$ where F^n denotes the n-fold convolution of F with itself. The renewal function $M(\cdot)$ is finite[6] and $M(x)$ can be interpreted as the expected number of jobs before the cumulative service times exceed x.

THE DERIVATION OF THE FORMULA FOR THE AVERAGE COST

We now consider the queueing system controlled by the D-policy. The state of the system can be described by a point in $\{x|x \text{ real}, 0\leq x\leq D\}\cup\{x'|x \text{ real}, x>0\}$ where state $x(x')$ corresponds to the situation where the workload equals x and the server is off(on). For any $t\geq 0$, let $X(t)$ be the state of the system at epoch t, and let $Z(t)$ be the total cost incurred during $[0,t]$.

We now introduce a Markov chain imbedded at points in time where either the server is turned off or is left off. To do this, assume that $X(0)=x$ for some $0\leq x\leq D$. Let $T_0=0$, and, for $n\geq 1$, let T_n be the nth epoch such that $X(T_n)\in\{x|0\leq x\leq D\}$ and $X(T_n) \neq X(T_n^-)$. For any $n\geq 0$, let $X_n = X(T_n)$. The process $\{X_n\}$ is a Markov chain with the state space $[0,D]$. Let $P_x(y)=\Pr\{X_n\leq y|X_{n-1}=x\}$. Then, $P_x(y)=F(y-x)+1-F(D-x)$ where $F(u)=0$ for $u<0$. Let $N=\inf\{n\geq 1|X_n=0\}$.

THEOREM 1. *(a) The Markov chain $\{X_n\}$ has a unique stationary probability distribution function $Q(\cdot)$ such that $Q(y)=\int_0^D P_x(y)dQ(x)$ for $0\leq y\leq D$.*

(b) $Q(y)=\{1+M(y)\}/\{1+M(D)\}$ for $0\leq y\leq D$.

(c) For initial state $X_0=0$, $\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n} Ef(X_k) = \int_0^D f(x)dQ(x)$ for any Baire function f such that $\int_0^D |f(x)|dQ(x)$ is finite.

PROOF. Clearly, $\Pr\{X_n=0 \text{ for some } n\geq 1|X_0=x\}=1$ and $E(N|X_0=x)<\infty$ for all $0\leq x\leq D$. The parts (a) and (c) now follow from Theorem 4 in the appendix. From part (a) we easily obtain $Q(y)=Q(0)+\int_0^y Q(y-x)dF(x)$ for $0\leq y\leq D$. By this renewal equation(e.g. p. 35 in Ross[6]), $Q(y)=Q(0)\{1+M(y)\}$ for $0\leq y\leq D$. Together this and $Q(D)=1$ imply part (b).

For any $n\geq 0$, let $\tau_n=T_{n+1}-T_n$, and let Z_n be the total cost incurred in

$[T_n, T_{n+1})$ (the cost K is included in Z_n when $X_n=0$). The distributions of τ_n and Z_n are determined by X_n. Let $\tau(x)=E(\tau_n|X_n=x)$, and let $\zeta(x)=E(Z_n|X_n=x)$ for $0\le x\le D$. Observe that $\tau(\cdot)$ and $\zeta(\cdot)$ are bounded where $\tau(x)\ge 1/\lambda$ for all x. The process $\{X(t)\}$ can be regarded as a semi-Markov process having a cost structure where X_n is the nth state of the process, τ_n is the length of the (n+1)st transition interval, and Z_n is the cost incurred during the (n+1)st transition interval. It is easy to verify that EN, ET and EZ(T) are finite when $X(0)=0$ where $T=\inf\{t>0|X(t)=0,\ X(t)\ne X(t^-)\}$. Now, by the proof of Theorem 7.5 in Ross[6],

$$\lim_{t\to\infty} EZ(t)/t = \sum_{n=0}^{\infty} EZ_n / \sum_{n=0}^{\infty} E\tau_n \qquad \text{when } X(0)=0.$$

It now follows from part (c) of Theorem 1 that for $X(0)=0$,

$$\lim_{t\to\infty} EZ(t)/t = \int_0^D \zeta(x)dQ(x) \Big/ \int_0^D \tau(x)dQ(x). \tag{2}$$

We now prove the next result(cf. p. 36 in Part *I* of De Leve[3]).

THEOREM 2.

$$\int_0^D \zeta(x)dQ(x)= \int_0^D \{k_1(x)-k_0(x)\}dQ(x), \quad \int_0^D \tau(x)dQ(x)= \int_0^D \{t_1(x)-t_0(x)\}dQ(x).$$

PROOF. By the definitions of the functions $k_0, k_1, t_0, t_1, \zeta$ and τ,

$$k_1(x)=\zeta(x)+\int_0^D k_0(y)dP_x(y), \quad t_1(x)=\tau(x)+\int_0^D t_0(y)dP_x(y) \quad \text{for } 0\le x\le D.$$

Integrating both sides of each of these relations with respect to $Q(x)$ and using part (a) of Theorem 1, we get the desired result.

Observe that the functions k_1-k_0 and t_1-t_0 do not depend on the D-policy as the functions ζ and τ do. We now give the formula for the average cost.

THEOREM 3. *For each initial state,* $\lim_{t\to\infty} EZ(t)/t$ *equals*

$$[2(1-\rho)]^{-1}h\lambda\mu^{(2)} + [1+M(D)]^{-1}\ [K\lambda(1-\rho)+hDM(D)-h\int_0^D M(x)dx]. \tag{3}$$

PROOF. For each initial state the process $\{X(t)\}$ will eventually enter state 0 with probability 1 and the expected costs incurred until the first return to state 0 are finite. Using this it is a simple matter to show that $\lim_{t\to\infty} EZ(t)/t$ is independent of the initial state(cf. p. 161 in Ross[6]). The Theorem now follows after some algebra from (1), (2), part (b) of Theorem 1, and Theorem 2.

Remarks. 1. For each initial state, $\lim_{t\to\infty} Z(t)/t = \lim_{t\to\infty} EZ(t)/t$ with probability 1(see Theorem 3.16 in Ross[6]).

2. The smallest value of D with $D + \int_0^D M(x)dx = K\lambda(1-\rho)/h$ minimizes (3).

3. The derivation above can be extended straightforwardly to cover the M/G/1 alternating priority queue with 1-jobs and 2-jobs where the server is turned off when the system is empty and the server is turned on when $(V_1,V_2)\epsilon R$ where V_i denotes the workload of i-jobs and R is a two-dimensional region such that $(x_1,x_2)\epsilon R$ implies $(y_1,y_2)\epsilon R$ when $y_i \geq x_i$ for i=1,2.

APPENDIX

Consider a Markov chain $X_0,X_1,X_2,\ldots$ with stationary transition probability function $P(\cdot,\cdot)$ on $(X,\mathcal{B})$ where X is a Borel set of a finite dimensional Euclidean space and $\mathcal{B}$ is the class of all Borel sets $A \subseteq X$. Let $P^n(\cdot,\cdot)$ be the n-step transition probability function, $n \geq 0$. That is, $P^n(x,A) = \Pr\{X_n \epsilon A | X_0 = x\}$. We make the following assumption.

Assumption. There is some state(say state s) such that

$$\Pr\{X_n = s \text{ for some } n \geq 1 | X_0 = x\} = 1 \qquad \text{for all } x \epsilon X, \tag{4}$$

$$E(N|X_0=s)<\infty \quad \text{where } N=\inf\{n\geq 1 | X_n = s\}. \tag{5}$$

THEOREM 4.*(a) For some measure* π, $\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n} P^k(x,A)=\pi(A)$ *for all* $x\epsilon X$

and $A\epsilon\mathcal{B}$.

(b) π *is the unique probability measure with* $\pi(A)=\int_X P(x,A)\pi(dx)$ *for all* $A\epsilon\mathcal{B}$.

(c) For initial state $X_0=s$, $\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n} Ef(X_k)=\int_X f(x)\pi(dx)$ *for any real-valued Baire function f such that* $\int_X |f(x)|\pi(dx)$ *is finite.*

PROOF. Let ${}_sP^n(x,A)=\Pr\{X_n\epsilon A,\ X_k\neq s \text{ for } 1\le k\le n|X_0=x\}$ for $n\ge 1$, and let ${}_sP^0(x,A)=P^0(x,A)$. Define $f_n(x)=\Pr\{N=n|X_0=x\}$ for $n\ge 1$, and let $f_0(x)=1$. Then (cf. p. 365 in Vol. 2 of FELLER[4]), for any x and A,

$$P^n(x,A) = {}_sP^n(x,A) + \sum_{k=0}^{n} P^{n-k}(s,A)f_k(x), \qquad n=0,1,\dots . \tag{6}$$

Since $\sum_0^\infty f_n(s)=1$, the relation (6) with x=s constitutes a renewal equation. Also, by (5), both $\sum nf_n(s)$ and $\sum {}_sP^n(s,A)$ are finite. Now, by applying the Key Renewal Theorem (see p. 292 in Vol. 1 of Feller[4]), for any $A\epsilon\mathcal{B}$,

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n} P^k(s,A) = \sum_{n=0}^{\infty} {}_sP^n(s,A)/ \sum_{n=0}^{\infty} nf_n(s). \tag{7}$$

For any $A\epsilon\mathcal{B}$, define $\pi(A)$ as the right side of (7). Clearly, π is a probability measure. By (4), $\sum_0^\infty f_n(x)=1$ and ${}_sP^n(x,A)\to 0$ as $n\to\infty$ for all x and A. Part (a) now follows from (6) and (7). Using part (a) it is easy to show that π satisfies the steady state equation in part (b) (cf. pp. 133-134 in BREIMAN[2]). Since the Markov chain $\{X_n\}$ has no two disjoint closed sets, π is the unique probability measure satisfying this equation (see Theorem 7.16 in Breiman[2]). To prove part (c), let m be a finite measure on $(X,\mathcal{B})$ such that $m(A)>0$ if and only if $s\epsilon A$. Then, by (4), $m(A)>0$ implies $\Pr\{X_n\epsilon A$ infinitely often$|X_0=x\}=1$ for all $x\epsilon X$, that is, $\{X_n\}$ satisfies the recurrence condition of Harris (cf. pp. 206-207 in JAIN[5]). Part (c) now follows from Theorem 3.3 in Jain[5].

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